INVARIANT SOLUTIONS OF THE NAVIER – STOKES EQUATIONS

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We examine certain invariant solutions of the Navier – Stokes equations. We prove theorems concerning the existence of solutions of boundary-value problems of the corresponding S/H systems.

1. It is well known that the widest group of continuous transformations admitting the system of Navier – Stokes equations

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - \Delta\mathbf{u} + \nabla\mathbf{p} = 0$$
$$\nabla\cdot\mathbf{u} = 0$$

is generated by the following operators:

$$X_0 = \frac{\partial}{\partial t}, \qquad S = \varphi(t)\frac{\partial}{\partial p}$$
$$X_{kl} = x^l\frac{\partial}{\partial x^k} - x^k\frac{\partial}{\partial x^l} + u^l\frac{\partial}{\partial u^k} - u^k\frac{\partial}{u^l} \quad \left(\begin{matrix} k<l \\ k, l = 1, 2, 3\end{matrix}\right)$$
$$T_k = \psi_k(t)\frac{\partial}{\partial x^k} + \psi_k'(t)\frac{\partial}{\partial u^k} - x_k\psi_k''(t)\frac{\partial}{\partial p} \quad (k = 1, 2, 3)$$
$$Z = 2t\frac{\partial}{\partial t} + \sum_{k=1}^{3}\left(x^k\frac{\partial}{\partial x^k} - u^k\frac{\partial}{\partial u^k}\right) - 2p\frac{\partial}{\partial p} \tag{1.1}$$

where φ, ψ_k (t) (k = 1, 2, 3) are arbitrary functions of the variable t.

2. In studying invariant solutions the most essential element is their interpretation. It is found that from the group G, generated by the operators (1.1), it is possible to select a subgroup G_{11} such that an arbitrary invariant solution constructed on its subgroups describes a flow with a free boundary. The following theorem, proved in [1], is valid.

THEOREM 1. If $u^k = \varphi^k$ (**x**, t) and N are invariant manifolds relative to one and the same subgroup H of the group G_{11}, then even the conditions on the free boundary

$$(-\mathbf{p}I + 2D)\nabla\mathbf{F} = 0, \qquad \mathbf{F}_t + \mathbf{u}\cdot\nabla\mathbf{F} = 0$$

are also invariant relative to this same subgroup [N : **F** (**x**, t) = 0 is the equation of the free boundary].

Therefore, from the point of view of applications to problems with a free boundary, there is interest in classifying dissimilar subgroups of the first, second, and thirdorders of the group G_{11}. We write out separately a basis of the group G_{11}

$$X_0, X_{kl}, Z, X_k = \frac{\partial}{\partial x^k}, \quad Y_k = t\frac{\partial}{\partial x^k} + \frac{\partial}{\partial u^k} \quad (k = 1, 2, 3)$$

Following a known method [2], Ovsyannikov constructed optimal systems of subgroups of the first, second, and third orders. In constructing optimal systems use was made of the fact that they are all solvable except for one, namely, $\langle X_{12}, X_{23}, X_{31}\rangle$. Therefore, if we know an optimal system θ_{s-1}, we can extend each (s-1)-dimensional subalgebra of the θ_{s-1} system to a subalgebra of dimensionality s and eliminate similar subalgebras of this dimensionality to obtain an optimal system θ_s of s-dimensional subalgebras [3].

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This was done in [1] in the case of two independent variables. As is to be expected, the majority of invariant solutions are already known and have been thoroughly studied [4]. We shall have something to say concerning some new invariant solutions later.

3. At the present time several examples are known of exact solutions describing the motion of a liquid with free boundaries (see, for example [1, 4, 5]). We give yet another example, a fairly simple one, of a flow with a free boundary.

Suppose that a viscous incompressible liquid occupies initially a spherical layer ($R_{20} \le r \le R_{10}$) and has a given radial speed. The motion is assumed to be spherically symmetric. The case $R_{10} = \infty$ was considered by Rayleigh [6]. From the Navier–Stokes equations, written in a spherical coordinate system, we obtain equations for v_r and p, which are to be solved in the domain $\Omega : \{t > 0,\ R_2(t) \le r \le R_1(t)\}$. Here $r = R_{1,2}(t)$ are, respectively, the outer and inner boundaries of the spherical layer, which are unknown beforehand. Integrating the continuity equation, we obtain

$$v_r = r^{-2}\varphi(t) \tag{3.1}$$

Equating the stress vector on the free boundary to zero, we obtain the following boundary conditions:

$$\tau_{rr} \equiv -p - 4\nu\varphi / r^3 = 0 \quad \text{for} \quad r = R_{1,2}(t) \tag{3.2}$$

From the kinematic condition on the free boundary

$$dR_{1,2}(t) / dt = \varphi / R_{1,2}^2(t) \tag{3.3}$$

we obtain the volume conservation law

$$R_1^3(t) - R_2^3(t) = R_{10}^3 - R_{20}^3 \equiv a^3 > 0 \tag{3.4}$$

Further,

$$d\varphi / dt = (d\varphi / dR_2)(dR_2 / dt)$$

so that the momentum equation reduces to

$$\frac{d\varphi}{dR_2} = \frac{\varphi}{2R_2}\left[1 + \frac{R_2}{R_1} + \frac{R_2^2}{R_1^2} + \frac{R_2^3}{R_1^3}\right] 4\nu - \left[1 + \frac{R_2}{R_1} + \frac{R_2^2}{R_1^2}\right] \tag{3.5}$$

To Eq. (3.5) we must adjoin the initial condition

$$\varphi(R_{20}) = \Phi_0 \tag{3.6}$$

The Cauchy problem (3.5), (3.6) may be solved explicitly:

$$\varphi = \left\{\Phi_0 - \int_{R_{20}}^{R_2(t)} g \exp\left[-\int_{R_{20}}^{R_2(t)} f dR_2\right] dR_2\right\} \exp\left\{\int_{R_{20}}^{R_2(t)} f dR_2\right\}$$

$$f \equiv \frac{1}{2R_2}\left[1 + \frac{R_2}{R_1} + \frac{R_2^2}{R_1^2} + \frac{R_2^3}{R_1^3}\right], \quad g \equiv 4\nu\left[1 + \frac{R_2}{R_1} + \frac{R_2^2}{R_1^2}\right]$$

To values of $\Phi_0 > 0$ there corresponds a divergence of the spherical layer while to values of $\Phi_0 < 0$ there corresponds a compression of this layer. It is clear that for a divergence of the spherical layer there always exists an $R_2 = R_*$, such that $\varphi(R_*) = 0$. We show that the time of divergence of the spherical layer to the critical radius R_* is infinite. Indeed, let $R_2 \to R_*$, then the first term on the right side of Eq. (3.5) tends to zero and the second tends towards some constant quantity. Therefore $\varphi = O(R_* - R_2)$ and the integral

$$\int_{R_{20}}^{R_2(t)} \frac{R_2^2}{\varphi(R_2)} dR_2 = t \tag{3.7}$$

diverges for $R_2 \to R_*$. Similarly, in the case of compression there always exists an $R_2 = R'_*$, such that $\varphi(R'_*) = 0$, and the time of compression of the spherical layer to the critical radius R'_* is infinite.

We consider yet another problem connected with a spherical layer. The statement of the problem differs from that of the preceding one in that we assume the difference of the pressures on the inner and outer boundaries of the spherical layer to be nonzero and, in fact, a function of the time. As before, the volume conservation law is satisfied, the only change being that in the equation for determining φ. The latter equation and the initial condition which φ satisfies are

$$\varphi \frac{d\varphi}{dR_2} - \frac{\varphi}{2} R_2^2 \left[\left(\frac{1}{R_2} + \frac{1}{R_1}\right)\left(\frac{1}{R_2^2} + \frac{1}{R_1^2}\right)\right] + 4\nu\varphi R_2^2 \left[\frac{1}{R_1^2} + \frac{1}{R_1 R_2} + \frac{1}{R_2^2}\right] - \frac{\psi R_1 R_2^3}{R_1 - R_2} = 0 \qquad (3.8)$$

$$\varphi(R_{20}) = \Phi_0 \qquad (3.9)$$

where $\psi = f_1(t) - f_2(t)$; $f_1(t)$ is the pressure on the inner boundary of the spherical layer, $f_2(t)$ is the pressure on the outer boundary, and $r = R_{1,2}(t)$ are, respectively, the outer and inner boundaries. Suppose, for definiteness, that $\psi \geq c > 0$; then, using differential inequalities of Chaplygin type, we obtain the following result, analogous to that in [5]. With compression of the spherical layer with a positive pressure drop ψ and $\Phi_0 < 0$, an $R_2 = R_*$ always exists such that $\varphi(R_*) = 0$ and the time for compression to the critical radius R_* is finite. Following this, the divergence of the spherical layer commences. Here, in contrast to the previous case, the critical radius does not exist.

4. The structure of infinitely dimensional Lie groups has been insufficiently studied and, in particular, it is not known how to enumerate the dissimilar subgroups of a given group. It will therefore be necessary to limit ourselves merely to the construction of several examples of the use of operators belonging to the group G.

We consider a three-parameter subgroup $-\langle T_1, T_2, T_3\rangle$. The functions ψ_k entering the operators T_k are fixed but arbitrary. Solutions of the Navier–Stokes equations are to be found in the following form:

$$u_1 = x_1\psi_1(t) + A(t), \quad u_2 = x_2\psi_2(t) + B(t), \quad u_3 = x_3\psi_3(t) + C(t)$$
$$P = -{}^1\!/_2\,[x_1^2(\psi_1' + \psi_1^2) + x_2^2(\psi_2' + \psi_2^2) + x_3^2(\psi_3' + \psi_3^2)] + \gamma(t)$$

Solutions of this type were employed by Hopf to construct an example of a nonunique solution of a Cauchy problem for the Navier–Stokes equations in the class of solutions with a linear growth of velocities and a quadratic pressure [7]. However, Hopf's example was, in fact, constructed for the Euler equations. We give an example of nonuniqueness of a solution of a Cauchy problem, which essentially takes into account the presence of the viscous terms in the Navier–Stokes equations. To this end we consider the subgroup (planar case) $\langle T_1 + T_2, S\rangle$, wherein $\psi_1 \equiv 1$ and ψ_2 is a fixed function of the variable t and of class C^3. The invariants of this subgroup are the following:

$$J_1 = \xi \equiv x_2 - \psi_2 x_1, \quad J_2 = t, \quad J_3 = u_1, \quad J_4 = u_2 - x_1\psi_2'(t)$$

Therefore we seek the solution in the form

$$u_1 = u(\xi, t), \quad u_2 = x_1\psi_2'(t) + v(\xi, t)$$
$$P = {}^1\!/_2 x_1^2\,\psi_2''(t)\,\psi_2(t) - x_1 x_2 \psi_2''(t) + \gamma(\xi, t) \qquad (4.1)$$

It should be noted that the given solution is partially invariant. Substituting Eqs. (4.1) into the Navier–Stokes equations, we obtain, after eliminating the function $\gamma(\xi, t)$, the heat conduction equation

$$\frac{\partial W}{\partial \tau} = \frac{\partial^2 W}{\partial \xi^2}$$
$$W = (1 + \psi_2^2)\,u - \xi\psi_2', \qquad \tau = \int (1 + \psi_2^2(t))\,dt$$

From the continuity equation it follows that $v = \psi_2 u$. Suppose that $\psi_2 \in C^3$ is such that $\psi_2(0) = \psi_2'(0) = \psi_2''(0) = 0$, then the problem, generalizing the known problem concerning the diffusion of a vortex layer, with the initial velocity field $u_1 = u(x_2)$, $u_2 = 0$, has an infinite set of solutions in the class of functions increasing linearly with respect to one of the spatial coordinates. We note that the known solution of the problem concerning the diffusion of a vortex layer, in which $u_2 \equiv 0$, $u_1 = u(x_2, t)$, is obtained when $\psi_2 \equiv 0$.

We give yet another example of the use of operators of the full group G. We consider the subgroup $\langle X_{12} + S\rangle$ (planar case). It is necessary to seek the invariant solution in the following form:

$$u_r = u(r, t), \quad u_\theta = v(r, t), \quad p = -\varphi(t)\,\theta + \gamma(r, t)$$

Suppose that $u \equiv 0$; we can then pose the following boundary-value problem. We are given the annular sector $\Omega : \{R_2 \leq r \leq R_1,\ 0 < \beta \leq \theta \leq \alpha < 2\pi\}$, where $r = R_{1,2}$ are solid walls. We are required to describe the motion of a viscous incompressible liquid, of Poiseuille flow type, in a curvilinear tube under the action of a nonstationary pressure drop. The equations and boundary conditions are the following:

$$\frac{\partial\Upsilon}{\partial r}=\frac{v^2}{r}$$

$$\frac{\partial v}{\partial t}=\frac{\varphi(t)}{r}+\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}$$

$$v|_{t=0}=v_0(r)$$

$$v|_{r=R_{1,2}}=0 \qquad (v_0(R_1)=v_0(R_2)=0)$$

The solution of this problem is as follows:

$$v(r,t)=v_0(r)+\sum_{i=1}^{\infty}a_i(t)\,w_i(r)$$

$$w_i(r)=Y_1(\lambda_i R_2)\,J_1(\lambda_i r)-J_1(\lambda_i R_2)\,Y_1(\lambda_i r)$$

J and Y are Bessel functions of the first and second kinds, and

$$a_i(t)=e^{-\lambda_i^2 t}\int_0^t F_i(t)\,e^{-\lambda_i^2 t}dt$$

$$F=\frac{\partial}{\partial r}\left(\frac{1}{}\frac{\partial}{\partial r}(rv_0(r)\right)+\frac{\varphi(t)}{r}$$

The F_i are the coefficients in the Fourier series expansion of the function F with respect to the w_i. The characteristic numbers λ_i are obtained from the relations

$$Y_1(\lambda_i R_2)\,J_1(\lambda_i R_1)-J_1(\lambda_i R_2)\,Y_1(\lambda_i R_1)=0$$

5. We consider the two-parameter subgroup $-\langle k\partial/\partial p-\partial/\partial\theta,\ \partial/\partial t\rangle$. We seek an invariant solution of the Navier–Stokes equations in the form

$$u_r=u(r,z),\quad u_\theta=v(r,z),\quad u_z=w(r,z),\quad p=-k\theta+p(r,z)$$

We assume that in the plane θ =const we are given a domain Ω with a boundary $\partial\Omega\in C^{2+\alpha}$, where Ω does not contain the coordinate origin. By rotating the domain Ω through an angle β less than $2\pi/k$ about the z axis, we obtain a curvilinear tube of constant section. Let R be the distance of a tube section to the coordinate origin. We wish to describe the motion of the viscous incompressible liquid inside the tube under the action of a constant pressure drop at its ends.

We write the Navier–Stokes equations for the vector function **V** with components u, v, w in a cylindrical coordinate system

$$u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}-\frac{v^2}{r}=-\frac{\partial p}{\partial r}+\nu\left[\frac{1}{}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{\partial^2 u}{\partial z^2}-\frac{u}{r^2}\right] \tag{5.1}$$

$$u\frac{\partial v}{\partial r}+w\frac{\partial v}{\partial z}+\frac{uv}{r}=\frac{k}{r}+\nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)+\frac{\partial^2 v}{\partial z^2}-\frac{v}{r^2}\right] \tag{5.2}$$

$$u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}=-\frac{\partial p}{\partial z}+\nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)+\frac{\partial^2 w}{\partial z^2}\right] \tag{5.3}$$

$$\frac{\partial}{\partial r}(ru)+\frac{\partial}{\partial z}(rw)=0 \tag{5.4}$$

Along with the Eqs. (5.1)-(5.4) we have the boundary condition

$$\mathbf{V}|_{\partial\Omega}=0 \tag{5.5}$$

To solve the boundary-value problem (5.1)-(5.5) we introduce the functional space $H(\Omega)$ whose elements are vector functions $\mathbf{V}(r,z)$ defined in Ω. The space $H(\Omega)$ is obtained by a completion of the set $\dot{J}(\Omega)$ of all infinitely differentiable, finite in Ω, solenoidal vector functions in the norm induced by the scalar product

$$[\mathbf{u},\mathbf{v}]=\int_\Omega \nabla\mathbf{u}\cdot\nabla\mathbf{v}\,r\,dr\,dz$$

Let $\Phi(\Phi_1,\Phi_2,\Phi_3)\in H(\Omega)$. We multiply Eqs. (5.1)-(5.3) respectively by Φ_1, Φ_2, Φ_3, add, and integrate over Ω:

$$\int_\Omega (\nu\nabla\mathbf{V}\cdot\nabla\Phi - \mathbf{V}\cdot\mathbf{V}\cdot\nabla\Phi)\, rdrdz = \int_\Omega \left[\frac{k}{r}\Phi_2 - \frac{v}{r}(\Phi_2 u - \Phi_1 v) - \frac{\nu}{r^2}(\Phi_1 u + \Phi_2 v)\right] rdrdz \tag{5.6}$$

The integral identity (5.6) serves as the basis for determining a generalized solution of the problem (5.1)-(5.5). We prove the following theorem.

<u>THEOREM 2</u>. The problem (5.1)-(5.5) has at least one generalized solution.

The proof of the theorem stating that a solution of the problem (5.1)-(5.5) exists is analogous to the prcof of the theorem concerning the existence of a solution of the nonlinear stationary interior boundary-value problem presented in [8]. Application of the method used in [8] is justified by the availability of the a priori estimate

$$\|\mathbf{V}\|_{H(\Omega')} \leqslant kC(\Omega') \tag{5.7}$$

The estimate (5.7) is obtained from Eq. (5.6) by putting $\Phi \equiv \mathrm{V}$ and making the change of variable

$$r = R + \xi$$

6. Consider the subgroup $\langle k\,\partial/\partial z - \partial/\partial\theta,\ \partial/\partial t\rangle$. Relative to this subgroup an invariant solution has the form

$$u_r = u(r,\xi),\quad u_\theta = v(r,\xi),\quad u_z = w(r,\xi),\quad p = p(r,\xi),\quad \xi = z + k\theta$$

Substituting these expressions into the Navier–Stokes equations, written in a cylindrical coordinate system, we obtain

$$u\frac{\partial u}{\partial r} + \left(\frac{1}{r}kv + w\right)\frac{\partial u}{\partial \xi} - \frac{v^2}{r} = -\frac{\partial p}{\partial r} + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \left(1 + \frac{k^2}{r^2}\right)\frac{\partial^2 u}{\partial \xi^2} - \frac{2k}{r^2}\frac{\partial v}{\partial \xi} - \frac{u}{r^2}\right] \tag{6.1}$$

$$u\frac{\partial v}{\partial r} + \left(\frac{1}{r}kv + w\right)\frac{\partial v}{\partial \xi} + \frac{uv}{r} = -\frac{k}{r}\frac{\partial p}{\partial \xi} + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \left(1 + \frac{k^2}{r^2}\right)\frac{\partial^2 v}{\partial \xi^2} + \frac{2k}{r^2}\frac{\partial u}{\partial \xi} - \frac{v}{r^2}\right] \tag{6.2}$$

$$u\frac{\partial w}{\partial r} + \left(\frac{1}{r}kv + w\right)\frac{\partial w}{\partial \xi} = -\frac{\partial p}{\partial \xi} + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \left(1 + \frac{k^2}{r^2}\right)\frac{\partial^2 w}{\partial \xi^2}\right] \tag{6.3}$$

$$\frac{\partial}{\partial r}(ru) + \frac{\partial}{\partial \xi}(kv + rw) = 0 \tag{6.4}$$

To the Eqs. (6.1)-(6.4) we adjoin the boundary condition

$$\mathbf{V}|_{\partial\Omega} = \mathbf{A}(r,\xi) \tag{6.5}$$

and the condition of periodicity in ξ with the period l

$$\mathbf{V}(r, \xi + l) = \mathbf{V}(r,\xi) \tag{6.6}$$

The domain Ω in the variables r and ξ represents the part of the plane bounded by the lines $\xi = 0$, l, $r = R_2$ and the curve $r = r(\xi)$ where r varies over the interval $R_1 \leq r \leq R_2$. The vector $\mathbf{A}(r, \xi)$ has the components $\{0, qr, q\}$, $\{0, qR_1, q\}$, $\{0, 0, 0\}$. The given spiral solution can be interpreted as the solution of the problem concerning "the spiral of Archimedes."

<u>THEOREM 3</u>. The problem (6.1)-(6.6) has at least one generalized solution.

We consider in the domain Ω the set $\dot{H}(\Omega, l)$ of infinitely differentiable solenoidal vectors such that they are periodic in ξ with period l and vanish near the boundary $\partial\Omega$. Let $H(\Omega, l)$ be the Hilbert space with the norm

$$\|\mathbf{u}\|^2_{H(\Omega,l)} = \int_\Omega \left\{\sum_{i=1}^{3}\left(\frac{\partial u_i}{\partial r}\right)^2 + \left(\frac{\partial u_i}{\partial \xi}\right)^2\right\} rdrd\xi \tag{6.7}$$

The space $H(\Omega, l)$ is obtained by a completion of $\dot{H}(\Omega, l)$ in the norm (6.7). Let $\Phi \in H(\Omega, l)$. We multiply Eqs. (6.1)-(6.3) respectively by φ_1, φ_2, φ_3, add, and integrate over Ω

$$\int_\Omega \{\nu\nabla\mathbf{V}\cdot\nabla\Phi - \mathbf{V}\cdot\mathbf{V}\cdot\nabla\Phi\}\, rdrd\xi = \int_\Omega \left\{\varphi_1\frac{v^2}{r} - \varphi_2\frac{uv}{r} - \nu\left[\frac{2k}{r^2}\left(\varphi_1\frac{\partial v}{\partial \xi} - \varphi_2\frac{\partial u}{\partial \xi}\right) + \varphi_1\frac{u}{r^2} + \varphi_2\frac{v}{r^2}\right]\right\} rdrd\xi \tag{6.8}$$

An arbitrary solution of the system (6.1)-(6.4) satisfies the integral identity (6.8) independently of the boundary conditions.

DEFINITION. A function $\mathbf{V}(r, \xi)$ is called a generalized solution of the problem (6.1)-(6.6) if there exists a solenoidal vector function $\mathbf{a} \in W_2^1(\Omega, l)$ such that $\mathbf{a}|_{\partial\Omega} = \mathbf{A}$, $\mathbf{V} - \mathbf{a} \equiv \mathbf{u} \in H(\Omega, l)$; the identity (6.8) is satisfied for arbitrary $\boldsymbol{\Phi} \in H(\Omega, l)$.

We can show that Eq. (6.8) is equivalent to an operator equation of the form $\mathbf{u} = \mathbf{F}\mathbf{u} + f$ with a completely continuous operator F, analogous to the procedure followed in [8]. Therefore the proof of the theorem concerning the existence of a solution reduces to obtaining an a priori estimate of $\|\mathbf{u}\|$ in the space $H(\Omega, l)$.

We remark that to obtain such an estimate is not a trivial matter. This is connected with the fact that the domain Ω in the variables r, θ, and z is unbounded. Therefore the estimate of the Dirichlet integral obtained in [8] for the interior stationary problem is not applicable here. However in the variables r and ξ the domain Ω is bounded, a fact which enables us to obtain an a priori estimate. To derive this a priori estimate we put $\mathbf{V} = \mathbf{a} + \mathbf{u}$, $\boldsymbol{\Phi} \equiv \mathbf{u}$, $\mathbf{u}\,(u_1, u_2, u_3)$ in the identity (6.8). Then we obtain the following equation:

$$\begin{aligned}
&\nu[\mathbf{u},\mathbf{u}] + \nu[\mathbf{a},\mathbf{u}] - \{\mathbf{u},\mathbf{u},\mathbf{u}\} - \{\mathbf{a},\mathbf{u},\mathbf{u}\} - \{\mathbf{u},\mathbf{a},\mathbf{u}\} - \{\mathbf{a},\mathbf{a},\mathbf{u}\} = \\
&= -\int_\Omega \frac{1}{r}\left(a_1 u_2^2 + a_1 a_2 u_2 - a_2 u_1 u_2 - a_2^2 u_1\right) r\,dr\,d\xi - \\
&- \nu \int_\Omega \frac{1}{r^2}\left(u_1^2 + u_2^2\right) r\,dr\,d\xi - \nu \int_\Omega \frac{1}{r^2}\left(a_1 u_1 + a_2 u_2\right) r\,dr\,d\xi - \\
&- 2\nu k \int_\Omega \frac{1}{r^2}\left(u_1 \frac{\partial u_2}{\partial \xi} - u_2 \frac{\partial u_1}{\partial \xi}\right) r\,dr\,d\xi - 2\nu k \int_\Omega \frac{1}{r^2}\left(u_1 \frac{\partial a_2}{\partial \xi} - u_2 \frac{\partial a_1}{\partial \xi}\right) r\,dr\,d\xi
\end{aligned}$$

where

$$[\mathbf{u},\mathbf{u}] = \int_\Omega \nabla\mathbf{u}\cdot\nabla\mathbf{u}\, r\,dr\,d\xi \equiv \int_\Omega \left\{\sum_{i=1}^{3}\left[\left(\frac{\partial u_i}{\partial r}\right)^2 + \left(1 + \frac{k^2}{r^2}\right)\left(\frac{\partial u_i}{\partial \xi}\right)^2\right]\right\} r\,dr\,d\xi$$

$$\{\mathbf{u},\mathbf{v},\mathbf{w}\} = \int_\Omega \mathbf{u}\cdot\mathbf{v}\cdot\nabla\mathbf{w}\, r\,dr\,d\xi, \quad \nabla\left(\frac{\partial}{\partial r}, \frac{k}{r}\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \xi}\right)$$

But since

$$\frac{1}{r^2}\left[\left(u_1^2 + u_2^2\right) + 2k\left(u_1 \frac{\partial u_2}{\partial \xi} - u_2 \frac{\partial u_1}{\partial \xi}\right)\right] = \frac{1}{r^2}\left\{\left(u_1 + k\frac{\partial u_2}{\partial \xi}\right)^2 + \left(u_2 - k\frac{\partial u_1}{\partial \xi}\right)^2\right\} - \frac{k^2}{r^2}\left[\left(\frac{\partial u_1}{\partial \xi}\right)^2 + \left(\frac{\partial u_2}{\partial \xi}\right)^2\right]$$

then, applying Hopf's lemma [9] with $\varepsilon = \nu/2\,(1 + C)$, where C is the constant from Poincare's equation [8], we obtain the required estimate

$$\|\mathbf{u}\|_{H(\Omega, l)} \leqslant \nu^{-1} C_0(\Omega, \mathbf{a}, k)$$

7. We consider the subgroup $\langle k\partial/\partial z - \partial/\partial\theta + \partial/\partial p, \partial/\partial t\rangle$. An invariant solution relative to this subgroup has the form

$$u_r = u(r, \xi), \; u_\theta = v(r, \xi), \quad u_z = w(r, \xi), \; p = k^{-1}z + p(r, \xi), \; \xi = z + k\theta$$

Substituting these expressions in the Navier–Stokes equations, written in a cylindrical coordinate system, we obtain

$$u\frac{\partial u}{\partial r} + \left(\frac{1}{r}kv + w\right)\frac{\partial u}{\partial \xi} - \frac{v^2}{r} = -\frac{\partial p}{\partial r} + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \left(1 + \frac{k^2}{r^2}\right)\frac{\partial^2 u}{\partial \xi^2} - \frac{2k}{r^2}\frac{\partial v}{\partial \xi} - \frac{u}{r^2}\right] \tag{7.1}$$

$$u\frac{\partial v}{\partial r} + \left(\frac{1}{r}kv + w\right)\frac{\partial v}{\partial \xi} + \frac{uv}{r} = -\frac{k}{r}\frac{\partial p}{\partial \xi} + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \left(1 + \frac{k^2}{r^2}\right)\frac{\partial^2 v}{\partial \xi^2} + \frac{2k}{r^2}\frac{\partial u}{\partial \xi} - \frac{v}{r^2}\right] \tag{7.2}$$

$$u\frac{\partial w}{\partial r} + \left(\frac{1}{r}kv + w\right)\frac{\partial w}{\partial \xi} = -\frac{\partial p}{\partial \xi} + \frac{1}{k} + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \left(1 + \frac{k^2}{r^2}\right)\frac{\partial^2 w}{\partial \xi^2}\right] \tag{7.3}$$

$$\frac{\partial}{\partial r}(ru) + \frac{\partial}{\partial \xi}(kv + rw) = 0 \tag{7.4}$$

The "no slip" condition yields the boundary condition

$$\mathbf{V}|_{\partial\Omega} = 0 \tag{7.5}$$

We seek a solution periodic in ξ with period l

$$\mathbf{V}(r, \xi + l) = \mathbf{V}(r; \xi) \tag{7.6}$$

The problem posed may be interpreted as the flow of a viscous incompressible liquid inside a coil.

Proceeding in the manner we did in the previous section, we obtain the integral identity

$$\int_\Omega \left\{\varphi_2 \frac{uv}{r} - \varphi_1 \frac{v^2}{r} + \nu\left[\frac{2k}{r^2}\left(\varphi_1 \frac{\partial v}{\partial \xi} - \varphi_2 \frac{\partial u}{\partial \xi}\right) + \frac{1}{r^2}(\varphi_1 u + \varphi_2 v)\right]\right\} r dr d\xi -$$
$$- \int_\Omega \mathbf{V}\cdot\mathbf{V}\cdot\nabla\mathbf{\Phi} r dr d\xi = -\nu \int_\Omega \nabla\mathbf{V}\cdot\nabla\mathbf{\Phi} r dr d\xi + \frac{1}{k}\int_\Omega \varphi_3 r dr d\xi, \tag{7.7}$$
$$\mathbf{\Phi}(\varphi_1, \varphi_2, \varphi_3) \in H(\Omega, l)$$

The identity (7.7) serves to determine the generalized solution of the problem (7.1)-(7.6).

THEOREM 4. The problem (7.1)-(7.6) has at least one generalized solution.

As before, it is sufficient to obtain an a priori estimate of $\|\mathbf{V}\|_{H(\Omega, l)}$. To obtain it we put $\mathbf{\Phi} \equiv \mathbf{V}$ in Eq. (7.7). We note that the integral

$$\int_\Omega \mathbf{V}\cdot\mathbf{V}\cdot\nabla\mathbf{V} r dr d\xi = 0$$

therefore the following equation is valid:

$$\nu \int_\Omega \left\{\sum_{i=1}^{3}\left[\left(\frac{\partial u_i}{\partial r}\right)^2 + \left(1 + \frac{k^2}{r^2}\right)\left(\frac{\partial u_i}{\partial \xi}\right)^2\right]\right\} r dr d\xi +$$
$$+ \nu \int_\Omega \left[\frac{1}{r^2}(u_1^2 + u_2^2) + \frac{2k}{r^2}\left(u_1 \frac{\partial u_2}{\partial \xi} - u_2 \frac{\partial u_1}{\partial \xi}\right)\right] r dr d\xi = \frac{1}{k}\int_\Omega u_3 r dr d\xi$$

We write the second integral on the left side of this equation in the form

$$\nu \int_\Omega \left\{\frac{1}{r^2}\left[\left(u_1 + k\frac{\partial u_2}{\partial \xi}\right)^2 + \left(u_2 - k\frac{\partial u_1}{\partial \xi}\right)^2\right] - \frac{k^2}{r^2}\left[\left(\frac{\partial u_1}{\partial \xi}\right)^2 + \left(\frac{\partial u_2}{\partial \xi}\right)^2\right]\right\} r dr d\xi$$

Cancelling off the common terms, we obtain the required estimate

$$\|\mathbf{V}\|_{H(\Omega, l)} \leqslant k^{-1} C(\Omega)$$

We remark that a uniqueness theorem holds for problems (5.1)-(5.5) and (7.1)-(7.6), but only for additional restrictions on the quantity k.

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